Semestral Exam
B.Math Algebra-IV
2015-2016

Time: 3 hrs
Max score: 100

Answer question (1) and any **five** from the rest.

(1) State true or false. Justify your answers with explanation.
(i) The extension $\mathbb{Q}(\sqrt[3]{2}, \sqrt{3})$ over $\mathbb{Q}$ is normal.
(ii) If the Galois group of a polynomial $f(x) \in \mathbb{Q}[x]$ has odd order, then all roots of $f(x)$ are real.
(iii) Let G be a finite group. Then there exists a field F and a polynomial $f(x) \in F[x]$ such that the Galois group of $f(x)$ is G.
(iv) No irreducible polynomial of degree 5 is solvable by radicals.
(v) Galois group of an irreducible cubic is either cyclic group of order 3 or is S_3. (5×5)

(2) (a) Determine the Galois group of the cyclotomic field $\mathbb{Q}(\zeta_n)$ over $\mathbb{Q}$, where ζ_n denotes a primitive n-th root of unity.
(b) Determine intermediate fields of the extension $\mathbb{Q}(\zeta_7)|\mathbb{Q}$.
(c) Show that each of these intermediate extensions are simple, by finding primitive elements. $(6 + 4 + 5)$

(3) (a) Define constructible numbers. When is a regular n-gon said to be constructible?
(b) Prove that a regular n-gon is constructible if and only if $\phi(n)$ is a power of 2. (ϕ denotes the Euler-phi function). $(3 + 12)$

(4) (a) Show that the finite field $\mathbb{F}_{p^n}$ is a simple extension of $\mathbb{F}_p$.
(b) Prove that the polynomial $x^{p^n} - x$ is the product of all distinct irreducible polynomials in $\mathbb{F}_p[x]$ of degree d, where d runs over all divisors of n. $(5 + 10)$

(5) (a)(i) Determine the splitting field K of the polynomial $x^p - x - a$ over $\mathbb{F}_p$, where $a \neq 0$, $a \in \mathbb{F}_p$.
(ii) Show that the extension $K|\mathbb{F}_p$ is a cyclic extension. $(7 + 8)$

Please turn over

(6) Let F be a field of characteristic not dividing n, and containing all nth-roots of unity.
(a) Show that if K is a cyclic extension of F of degree n, then $K = F(\sqrt[n]{a})$ for some $a \in F$.
(b) Conversely, show that if $L = F(\sqrt[n]{a})$ for some $a \in F$, then L is a cyclic extension of F of degree dividing n. $(9 + 6)$

(7) Consider the polynomial $f(x) = x^5 - 4x + 2$ over $\mathbb{Q}$.
(a) Show that $f(x)$ is irreducible and has 3 real roots and 2 complex roots.
(b) Hence show that f(x) is not solvable by radicals. $(8 + 7)$

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